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ABSTRACT

Technological advances during the past few decades have revolutionized many complex systems that influence human activity. As the rate of technological progress accelerates, these systems will become more complex, and new ones will evolve. Citizens in a technological society need to be able to make intelligent choices about how technology will affect their lives. This requires a better understanding of the technology itself; its potential and its limitations. This is a module for a course designed to introduce students, particularly those not majoring in the sciences or mathematics, to technological concepts. In this module, exponential growth is defined qualitatively, graphically, and algebraically. The module presents a range of applications, from those familiar to the student to those basic to technological understanding. This unit is appropriate for an introductory course in quantitative methods or technology studies. (CW)



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# Exponential Growth

by Morton Tavel

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*Designed to further student understanding of technological issues, this module is one in a series developed in a project on "Technology and the Liberal Arts," which is sponsored by CIC—the Council of Independent Colleges, Washington, D.C.*

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Technological advances in just the past few decades have revolutionized many of the complex systems that guide our lives—the way we travel, for example, and how we communicate with one another, heal our sick, even the way we grow our food. As the rate of technological progress accelerates, these systems will become more complex, and new ones will evolve. Technology will continue to be a preeminent force in our lives.

As citizens in a technological society, therefore, we need to be able to make intelligent choices about how technology will affect our future lives. This requires that we have a better understanding of technology itself. We need to understand its potential and limitations. Further, we need to understand the very *process* of technology, and have a sense of its methodology and rules.

The Council of Independent Colleges has been deeply involved in these issues since 1984, in a project entitled "Technology and the Liberal Arts." Working with faculty from small liberal arts colleges, CIC has sought to integrate the study of technology and technological issues into the liberal arts curriculum. Our goal is to improve the understanding of technology on the part of undergraduate students—the future citizens of the technological society.

This course module is one in a series developed in the CIC project under a grant from the Exxon Education Foundation. Each module is designed to introduce students, particularly those not majoring in the sciences, to a technological issue or concept. In the process of completing the exercises in the modules, we also hope students will develop an idea of the methodology of technological inquiry. Our goal in publishing the modules is to make teaching materials available for use in courses across the liberal arts disciplines. We are deeply indebted to the Exxon Education Foundation and its executive director, Arnold R. Shore, for supporting this endeavor.

I want to also acknowledge the many college faculty who have participated in CIC's technology project. To each, I offer my sincere thanks for pioneering the important effort to bring technology into the mainstream of the curriculum.

Allen P. Splete
President
The Council of Independent Colleges
October 1987

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# Exponential Growth

by Morton Tavel

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## INTRODUCTION

We live in a time when the entire world, and our society in particular, is preoccupied with growth. All the developing nations are concerned about the rates at which their populations are growing and, perhaps to a lesser extent, the rates at which the populations of other nations are growing. Industrialized nations are concerned about negative growth, specifically, the depletion of renewable and non-renewable resources. The availability of these commodities seriously affects the growth of their economies. There is also a general global concern over the growth in the production of waste products. Solid wastes are being produced in increasing amounts and the sites at which they can be safely disposed are often being filled up. The production of certain atmospheric pollutants has also increased, which, in turn, causes other growth problems such as the depletion of the ozone layer.

In this country we need only look at today's newspaper to be reminded of growth. There is the increase in the number of people that are affected by certain diseases, the growth of our national debt, the growth of our gross national product and the growth (or decline) in the use of certain technologies such as the computer or the automobile. Meanwhile, the ubiquitous bank advertisements herald new ways to make your money grow at astounding rates. We are clearly a nation of competing philosophies, where, simultaneously, bigger is better and small is beautiful.

In an era so marked by the importance of growth, it is critical that we understand all of its implications. We need to know precisely which things are growing larger and which are growing smaller. Equally, we must know

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which things are growing rapidly and how rapidly they are growing. Which are growing slowly? Which things are being used up faster than they can be replaced and which things cannot ever be replaced? To what extent does the growth of some commodity require or depend on the growth of another commodity? To what extent does the growth of a commodity affect or cause the growth of another? How is the quality of our lives dependent on the growth or depletion of these things? Not only will our understanding of the processes of growth give us some degree of control over our own future well being, but it will give us a means to judge the claims of others which are based on their supposed knowledge of growth.

If the growing quantity affects the quality of our lives, knowledge can be a prerequisite to action. Responsible urban planning, for example, requires dependable predictions of population changes, changes in traffic patterns and waste disposal needs. Some of these predictions must be quite detailed. If the population is increasing among the elderly, plans must be made for specific kinds of medical care, activities for retired people, social welfare, etc. If it is the teenage population that is increasing, job training programs must be installed, family planning programs must be activated, etc. It is clear that designing methods for accurately predicting patterns of growth must become a vital part of the technology of today.

How can we predict the manner in which some resource or commodity will grow or diminish in size? Somehow, we must make use of the cumulative knowledge we have up to the present time to make a reasonable projection about the future. There are several ways that this can be accomplished, the most accurate being by making use of a "model" of the growth process under consideration. The remainder of this module will be devoted to a thorough discussion of one particular growth model of amazingly broad applicability, Exponential Growth.

## THE USE OF MODELS

Science is a method of understanding the behavior of certain kinds of processes and mathematics is one of its most important tools. This is not meant to imply that mathematics is the only road to understanding, but it does provide a uniquely effective framework for dealing with quantities that must be described with precision and exactness— and these are just the things that science deals with.

In order to make the most efficient use of mathematics, scientists and engineers develop and use "models," ways of thinking about complex processes that strip away non-essentials and leave behind an essence that can be expressed in mathematical terms. Very often these models enable us to examine phenomena that appear, on the surface, to be vastly dissimilar and to find, beneath the surface, the operation of processes that are actually very similar. Of course, all models eventually outlive their usefulness and are replaced by other models. The particular model we will spend some time

discussing in this module is called "exponential growth".

## PROPERTIES OF GROWTH MODELS

So many different quantities grow in one way or another, that unless we can develop a simple model to describe and understand the process, we will be forever bogged down in the seemingly endless details of *what* is growing rather than *how* it is growing. Our model will assume that a large and important class of growing things have their growth completely described in terms of just a few simple mathematical quantities. Once these quantities are determined, either by observation of the growing object, or by the use of theories about the reasons for its growth, we can describe that growth very precisely and make accurate predictions about it. The model, however, will not necessarily explain the underlying reasons why the growth occurs the way it does. Basically, the exponential growth model is meant to describe growth processes, not explain them. Money in the bank grows for very different reasons than algae in a pond. It is precisely the differences in those reasons that produce the two very different disciplines of economics and biology. The descriptions of money growth and algae growth, however, are given by the same model; that is what makes it such an important tool. The model does not care whether it is being used by an economist or a biologist, although each of these users must be convinced that the model is appropriate to the subject of their study. That knowledge requires either careful observation, as in the case of the algae, or an understanding of how the system really operates, as is most appropriate for the bank account. In addition, the biologist and economist will almost certainly make different use of the predictions of the model. One may suggest an algaecide, the other a different bank.

In our model, growth will always be described as being relative to an underlying variable, which mathematicians would call the "independent variable." While the most common variable is time, things can grow relative to space and possibly other variables as well. In what follows, we will think of time as the independent variable, but this is only for convenience and to avoid a constant usage of the phrase, "the underlying variable".

A major characteristic of growth is its *rate*. The *rate of growth* of a quantity is the amount by which it changes in some unit of time. The rate of growth of your bank account might be in units of dollars per year. Rates of growth can be positive or negative, although growth that is characterized by a negative growth rate is more usually termed "depletion" rather than growth. Since so many different phenomena are described by the same model, it is not unusual to see different terminology. You must always pay careful attention to the description of the process, however, to be sure you know what it is that is growing. In many of the examples we will discuss it is the amount of a commodity that is growing. Money and population are two such commodities. In other examples it may be the consumption of a quantity that is growing. Oil and other mineral resources are examples of quanti-

ties in whose consumption growth rate we are interested .

If the growth of a quantity is carefully monitored by measuring the amount of that quantity at a succession of different times, it is possible to plot a graph of the various amounts (on the y-axis) vs. the times (on the x-axis) at which those amounts are present. Such a graph is called a *growth curve*. A growth curve can either be obtained experimentally, by actually measuring the growing quantity, or theoretically, by using results of the model.

## LINEAR GROWTH

The simplest growth model results when the growth rate is *constant;* when the quantity increases by the same amount in equal units of time. An example of such a growth process would be an intelligent diet. If a person diets so that he or she loses one pound every week, we have growth (depletion) with a negative rate of minus one pound per week. A growth process with a constant rate is called *linear growth,* because the shape of the growth curve is a straight line, whose slope is the rate. Linear growth is also referred to as *arithmetic growth*, from the analogy to what mathematicians call an arithmetic series, a succession of numbers that differ by a constant, eg. 1, 2, 3, 4...

The growth curve of the weight of a person who begins a diet weighing 190 lbs and takes off 1 pound per week for 10 weeks looks like the graph in Fig. 1.

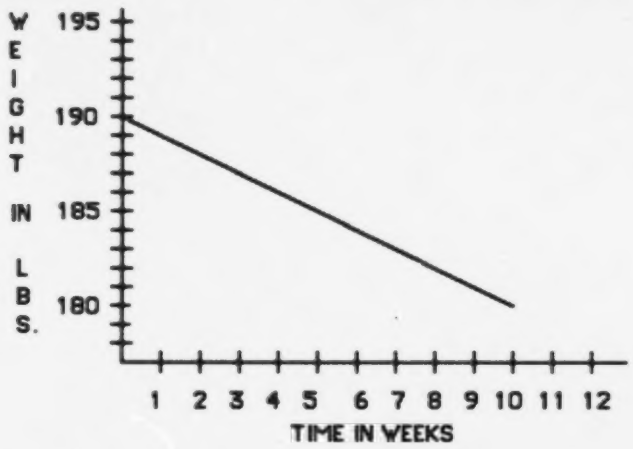



Fig. 1

The same growth curve can be described by an equation, in which W (t) denotes the dieter's weight at a time t, W(0) is the dieter's weight at time "zero," when the dieter begins the diet and R is the rate at which the dieter loses weight . The equation is:

$W(t) = W(0) - Rt$

To obtain the growth curve of Fig. 1, insert the values 190 lbs for W(0)

and 1 lb/week for R.

Another, somewhat more involved example of linear growth might be the buildup of silt in the bed of a river. If the silt is carried into the river by the runoff of rainfall on the surrounding land and if the amount of rainfall is fairly constant each year and if the amount of exposed soil remains fairly constant each year and, finally, if the flow rate of the river does not change significantly, then the amount of silt deposited on the river bed per year is also a constant. The growth of silt (i.e. its thickness) can then be described by a linear growth model.

**PROBLEM 1**: In terms of all the quantities described above, set up an equation for the increase of the silt layer on a river bed. Assume the river has a negligible flow. If you can do this, think about how the flow rate of the river would affect the deposition of silt.

## EXPONENTIAL GROWTH

Although linear growth is a valid model for certain processes that occur in nature, there are many processes that it does not accurately describe. This brings us to another and even more useful model, *exponential growth.* Exponential growth occurs when the growth rate of a quantity is not simply a constant, but is a *constant fraction of the quantity itself.* Thus, for exponential growth to occur, in each given unit of time the quantity must increase by a fraction of the amount already present. Recall that in linear growth the amount added each period was independent of the amount already there; in exponential growth, it depends very much on how much is already there. The more you have already, the more you get. It is a mathematical equivalent of the aphorism, "the rich get richer."

The growth that is described by this model is called exponential because the quantity increases as a power of the time. It is also sometimes called *geometric growth*, by analogy with the sequence of numbers that mathematicians call a *geometric series,* like 1, 2, 4, 8, ... , in which each new number is the same fixed multiple of the preceeding number (in this case the multiple is 2 ).

Let us now compare linear growth to exponential growth with a simple economic example, the growth of money in an interest-bearing bank account. Consider two different banks. The First National Linear Bank adds $5.00 to your account at the end of every year, regardless of how much is in the account. The Amalgamated Exponential bank gives 5% interest on your yearly balance at the end of every year. We will start out with $100.00 in each account, make no additional deposits and watch the balances grow. The table below compares your balance at the two banks over a period of 70 years.

| YEAR | LINEAR BANK BALANCE | EXPONENTIAL BANK BALANCE |
|---|---|---|
| 0 | $100.00 | $100.00 |
| 1 | 105.00 | 105.00 |
| 2 | 110.00 | 110.25 |
| 3 | 115.00 | 115.76 |
| 4 | 120.00 | 121.55 |
| 10 | 150.00 | 162.88 |
| 15 | 175.00 | 207.89 |
| 28 | 240.00 | 392.01 |
| 70 | 450.00 | 3,042.65 |

Clearly, your bank balance at Amalgamated Exponential is growing much more rapidly than that at First National Linear. We call Exponential's method of giving interest: "5% interest compounded annually." At the end of the first year you have:

$(1 + 0.05)\ \$100.00 = \$105.00$

At the end of the second year you have:

$(1 + 0.05)\ \$105.00 = \$110.25.$

This can also be written:

$(1 + 0.05)(1 + 0.05)\ \$100.00 = (1 + 0.05)^2\ \$100.00.$

The amount in your account after 70 years is:

$(1 + 0.05)^{70}\ \$100.00 = \$3,042.65$

In equation form: your yearly balance after t years, B(t), in terms of your initial balance, B(0), for annually compounded interest is:

$B(t) = (1 + \text{Rate})^t\ B(0)$

Suppose the bank chose to compound your interest quarterly, i.e. to give you 1/4 of the rate, four times a year. After the first quarter you would have:

$(1 + 0.05/4)\ \$100.00 = \$101.25$

After one year:

$( 1 + 0.05/4 )^4 \$100.00 = \$105.09$

These amounts are slightly more than you would have at the equivalent times under annual compounding. The bank might even go so far as to compound the interest daily, in which case your balance after one year would be:

$( 1 + 0.05/365 )^{365} \$100.00$

**PROBLEM 2:** Calculate the quantity above and compare it with $105.00

All of the above methods of compounding interest are exponential growth models, but they have one limiting characteristic: they are *discrete*. This means they only give you an answer at a specific time. If your bank compounds interest annually and you wish to withdraw your funds eleven months into the year, will you get any interest? Not if the bank sticks to the model. In the next paragraph we will introduce the very important *continuous model* of exponential growth. With this model, a bank could compute the interest it owes you at any time t and your interest is actually compounded continuously from day of deposit to day of withdrawal. Another nice feature of the continuous model is that it can be graphed as a continuous line, whereas the discrete models are just a sequence of points at the specific times the interest is compounded.

To derive the continuous model, we imagine that interest will be compounded on a time scale that is infinitesimally fine, i.e. by dividing the year into n parts, where n is a number that can be made infinitely large. The idea of dividing a time into infinitely many infinitesimally small pieces is a mathematical idealization, but it works rather well. If we do this, the amount of interest earned at any time t is:

$( 1 + 0.05/n )^{nt} \$100.00.$

The quantity $( 1 + 0.05/n )^{nt}$, as n approaches infinity, can be shown to be equal to $e^{0.05t}$, where e is the base of the natural logarithms (an infinite decimal which approximately equals 2.718). Note that in the above expression, t must be given in terms of years; thus, t= 1 day would be t = 1/365 years. We will use this expression for continuously compounded interest to plot two growth curves in Fig. 2 for the interest given by Amalgamated Exponential and First National Linear banks.

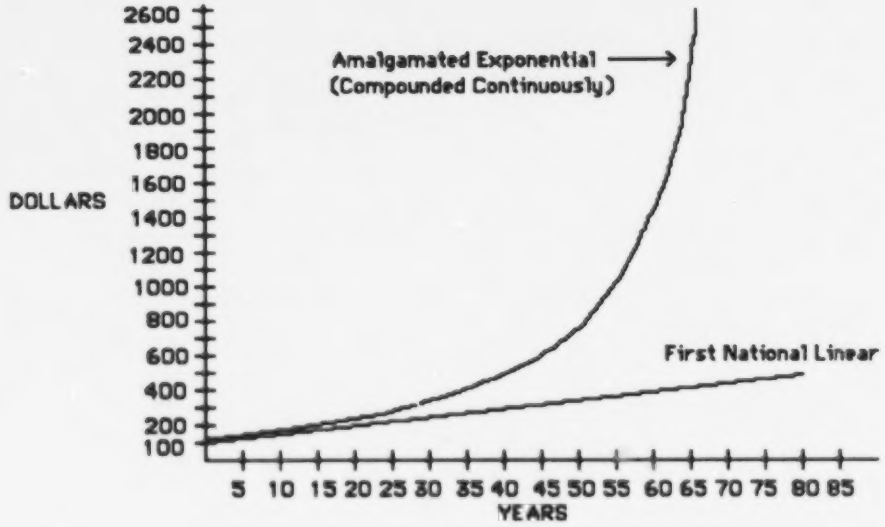



Fig. 2

The two growth curves in Fig. 2 vividly show the difference between linear and continuous exponential growth. It is also instructive to write down the equations that describe the linear and the exponential growths, from which the curves can be obtained. The symbols will be B(t), balance at time t; B(0), initial balance ($100.00 in this case); L, linear growth rate ($5.00 per year) and R, exponential growth rate, .05 per year.

Linear model: $B(t) = B(0) + Lt$

$B(t) = \$100.00 + (\$5.00)t$

Continuous Exponential model: $B(t) = B(0)\, e^{Rt}$

$B(t) = \$100.00\, e^{(.05)t}$

## THE IRA ACCOUNT

In the previous section we have examined the growth of an initial deposit in a bank account. We have seen that such a deposit grows exponentially if it accumulates compounded interest. We might also ask how the money in a bank account grows if additional interest earning deposits are made along with the money that is already there? An excellent example of this kind of growth is the IRA account, a tax-deferred Individual Retirement Account into which a single taxpayer may invest up to $2000 per year. Many banks and stock brokerages offer the investor accounts that meet government requirements for an IRA. In an attempt to attract investors, these firms advertise

their accounts by making claims that a person who opens up an IRA when 25 years of age and contributes the maximum $2000 to the account each year, will be a millionaire by retirement.

It turns out that the growth of an account to which additional deposits are made at a regular rate does have exponential characteristics, but it cannot be described by as simple an exponential growth model as we have been using. It cannot be described as some initial amount of money multiplied by a single exponential factor. Consider, for example, an IRA account that has been active for 10 years, earning 10% interest compounded continuously, to which $2000 has been added on Jan. 1 of each year. How much money will be in the account at the end of the 10th year? We can analyze the growth one deposit at a time. The first $2000 stays in the account for a full 10 years and grows to:

$$\$2000\, e^{(.1 \text{ per year})(10 \text{ years})} = \$2000\, e^{1} = \$5420$$

The second deposit grows for only 9 years at the same 10% interest rate, to become:

$$\$2000\, e^{(.1 \text{ per year})(9 \text{ years})} = \$2000\, e^{.9} = \$4920$$

It can be seen that the 10 deposits will finally yield:

$$\$2000\,[\, e^{1} + e^{.9} + e^{.8} + e^{.7} + e^{.6} + e^{.5} + e^{.4} + e^{.3} + e^{.2} + e^{.1}\,]$$

Adding up all these exponentials gives a final IRA account total of $35,146.00 for the 10 deposits of $2000 that were made during the 10 year period. Note that if the full $20,000 had been deposited as one lump in the account to earn interest for the full 10 years, the final amount would have been $54,200.00.

Professor Michael Olinick of Middlebury College has been kind enough to send me a simple computer program, written in BASIC, to calculate the growth of an IRA account for arbitrary times and interest rates. This program compounds the interest annually rather than continuously. I include this program in Appendix 1(page 39), so the computer oriented student can apply it to plot out his/her financial future. If you are able to run this program, be sure to find out if you will be a millionaire when you retire, given the present rates of interest.

## DOUBLING TIME

Exponential growth produces extremely large increases (or decreases) in a quantity in surprisingly short intervals of time. In fact, a unique characteristic of both the discrete and continuous models of exponential growth is *doubling time*, a fixed period of elapsed time ( or whatever the underlying variable is) during which the quantity doubles in amount. As we shall see in a

later section, if the quantity is *diminishing* exponentially in time we apply the term *half life*. We would use doubling length if the quantity grew in space rather than time.

A simple way to see the extraordinary growth properties of a model that produces a continual doubling is to consider this old Persian fable. A very clever and able craftsman (craftsperson?) once presented a beautiful chessboard to the king of Persia. The king was so taken by the beauty of the gift that he told the craftsman he could have in return anything reasonable that he chose. The craftsman, claiming that his needs were simple, asked the king to place 1 grain of rice on the first square of the chessboard, 2 grains on the second square and to continue doubling the amount on each succeeding square until all 64 squares were covered. The king felt this was a very modest request and immediately agreed to grant it. Little did the king realize that he had effectively agreed to give the craftsman his entire kingdom. The following table will show exactly how the total quantity of rice grows as the king doubles the amount in each succeeding square.

**FILLING SQUARES ON A CHESSBOARD**

| Square Number | Grains On Square | Total |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 3 |
| 3 | 4 | 7 |
| 4 | 8 | 15 |
| 5 | 16 | 31 |
| 6 | 32 | 63 |
| 7 | 64 | 127 |
| 8 | 128 | 255 (1 tsp) |
| — | — | — |
| 13 | 4,096 | 8,191 (1 cup) |
| 14 | 8,192 | 16,385 (2 cups) |
| — | — | — |
| 16 | 32,768 | 65,535 |
| 17 | 65,536 | 131,071 |
| — | — | — |
| 32 | 2,147,000,000 | 4,295,000,000 (1 ton) |
| 33 | 4,295,000,000 | 8,590,000,000 (2 ton) |
| — | — | — |
| 64 | 9.223 ($10^{18}$) | 1.844 ($10^{19}$) (10 billion tons!) |

Another illustration of the frightening speed with which exponential growth creeps up on you appears in the book, "Limits to Growth" as a riddle. Suppose you own a pond on which a few lillies are growing. You notice that each day the lillies double in size. You are told by a knowledgeable friend that the lillies will cover the entire surface of the pond in just 30 days, so you had better begin to clear them away. Being a bit of a procrastinator, you

decide that you don't have to take any action until the pond is half covered. When does that happen? In 29 days. How long do you have to clear the pond? Just one day.

One of the fascinating features of exponential growth is that the amount of new material added in any doubling period is approximately equal to all the material that has been produced up to that period. This can be seen clearly by referring to the chessboard example. At each square the king had to give the craftsman more new rice than he had already given in all of the preceding squares. The king placed 4 grains in the third square, but he had only given 1 grain + 2 grains = 3 grains in the first two squares. The king places 8 grains in the fourth square, but he has only given 1 grain + 2 grains + 4 grains = 7 grains in the first three squares.

There are, as we have now seen, two equivalent ways of defining exponential growth.

1. A quantity, Q, grows exponentially if it increases by a constant fraction, R, of itself (R is the growth rate ) in each unit of time (or other appropriate variable). If the unit of time can be made infinitesimally small, we obtain a continuous model of exponential growth, $Q(t) = Q_0 e^{Rt}$.

2. A quantity grows exponentially if it doubles in amount with every passage of a certain finite period of time (or other appropriate variable) called the doubling period. In the fable of the chessboard the doubling period was a square. The amount of rice doubled at every successive square. In the riddle of the water lillies, the doubling period was one day.

There is an important yet quite simple approximate formula that should be committed to memory. It relates the doubling time, symbolized $T_2$, to the growth rate, R, in the continuous growth model:

$$T_2 = 0.7/R$$

In this formula R is the growth rate in units of "per unit time" or per unit other variable. If R is given as a percent rather than a fraction, the formula reads:

$$T_2 = 70/R(\text{in percent}).$$

There is really no mystery to the derivation of this formula; it is obtained by requiring that $T_2$ be the time at which the exponential would be exactly 2 :

$$e^{RT_2} = 2$$

Taking logarithms of both sides:

$\ln ( e^{RT_2} ) = \ln (2)$ (the natural logarithm of 2 is 0.69)
$RT_2 = 0.69$
$T_2 = 0.69/R$
In most instances 0.69 can be very well approximated by 0.7 and since 0.7 is easier to remember than 0.69, I prefer to use $T_2 = 0.7/R$ .

If the doubling property of exponential growth is to be stressed, it is more sensible to write the continuous growth model in the form:

$$Q(t) = Q_0 \, 2^{t/T_2}$$

This form is obtained by the following manipulation of exponentials.

$Q(t) = Q_0 \, e^{Rt}$
$Q(t) = Q_0 \, e^{(0.7/T_2)t}$
In the last step we have expressed R in terms of $T_2$.
But:

$$e^{(0.7/T_2)t} = e^{(0.7)(t/T_2)} = ( e^{0.7} )^{t/T_2}$$

The last step follows from the fact that e raised to a product of exponentials is the same as e raised to one factor of the product, then raised again to the other factor. Finally, since:

$e^{0.7} = 2$ , remember, we have approximated ln (2 ) by 0.7 ! We have:

$$e^{Rt} = 2^{(t/T_2)}$$

Returning to the exponential and linear bank account growth curves we can now see that the account in Amalgamated Exponential has a doubling time as indicated in Fig. 3.

$T_2 = 70/(5\%$ per year) = 14 years.

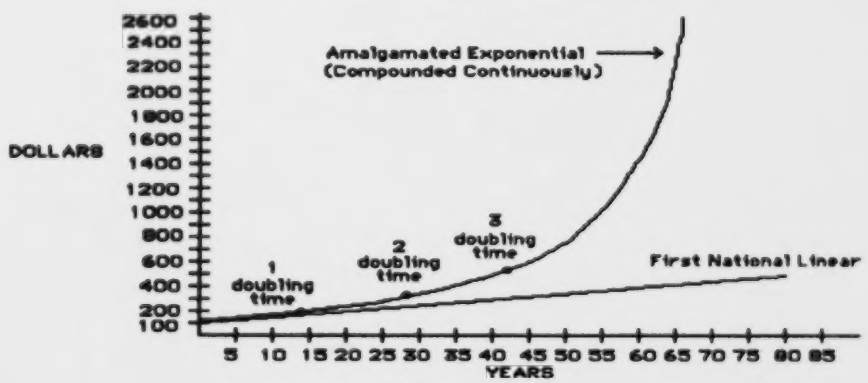



Fig. 3

The First National Linear bank will also eventually double your money. In fact, the initial $100.00 deposit will increase to $200.00 in exactly 20 years (20 years x $5.00 per year = $100.00). There is, however, no constant doubling time, since it would then take an additional 40 years to double the $200.00.

**PROBLEM 3:** How much money does Amalgamated Exponential add to your account during the fifth doubling time? What is the total amount they have put into your account during the first four doubling times.

**PROBLEM 4:** Although "doubling" has a good intuitive feel to it, which is why it is used, we could equally well associate a "tripling time" to exponential growth. What is the relation between tripling time, $T_3$, and the doubling time?

## THE MALTHUSIAN DOCTRINE

Thomas Robert Malthus (1766-1834), an English economist, was one of the first people to predict the prospect of dire consequences for the world because of the fact that linear growth could never keep pace with exponential growth. In an essay entitled "An Essay on the Principle of Population As It Affects the Future Improvement of Society", which he published anonymously in 1798, Malthus stated clearly and simply the following two observations.

1. "It may safely be pronounced, therefore, that population, when unchecked, goes on doubling itself every twenty-five years, or increases in a geometrical ratio."

2. ". . . the subsistence for man which the earth affords might be increased every twenty-five years by a quantity equal to what it at present produces."

In short, according to Malthus, population growth is exponential, whereas the growth of the food supply is, at best, linear ( increasing every 25 years by the amount it was producing in Malthus' time) and the latter simply cannot keep pace with the former. Malthus went on to claim that in two centuries the ratio of population to food supply would be 256/9.

**PROBLEM 5:** Show that, with Malthus' assumption of a 25 year doubling time, the ratio of population to food supply would indeed be 256/9 in two centuries.

Malthus' purpose in writing his essay was to warn the world of this striking and, in his opinion, continual conflict between population increase resulting from man's desire to procreate and the much more limited capacity of the earth to provide sustenance. Malthus' conclusions created an extraordinary

furor because they came at a time when it was widely believed that mankind had nothing but glorious times to look forward to. Nevertheless, we must now view his work with importance, if only for the fact that he attempted to combine sound mathematical reasoning with a strong sense of social consciousness. In addition, Malthus' notion of a continual "struggle" for existence had a great impact on Charles Darwin. Today, after many years of neglect and rejection, the Malthusian Doctrine, as it is called, is once again coming back into vogue.

**PROBLEM 6:** Recent estimates show that world food production is holding steady on a per capita basis. How does this compare with Malthus' prediction? Can you advance any reasons for this? Do you think this invalidates Malthus' work?

## GROWTH OF THE WORLD POPULATION

The growth of the world's population has been shown to follow an exponential growth model quite well over fairly long periods of time. This seems reasonable, since we know that the number of children produced in a unit of time should depend roughly on the number of adults of childbearing age. Similarly, the number of deaths per unit time should also depend on the size of the population. The increase in population at any given time should, therefore, be describable in terms of the product of some growth rate and the population at that time. People who study population growth, called demographers, define an overall *rate of natural increase* of population as the difference between the rate at which people are added to the population by births (called the *crude birth rate*) and the rate at which they are removed from the population by deaths ( called the *crude death rate)*. More sophisticated models include factors such as immigration and emigration, but these factors generally influence only a few special countries. The use of the rate of natural increase as the growth rate in an exponential growth model has proved to give a very adequate description of the growth of the world's population over a very long period of time.

The growth of the populations of individual countries also follow the exponential model. Not only does each country have its own growth rate, but, moreover, these rates tend to change with time as a result of complex social and economic factors. Because of these variations, long range predictions based on the exponential growth model may be subject to question, but short term predictions are not unreasonable.

Studies of population growth made during the period between 1950-1980 indicate that exponential growth is a good model to use if the period is broken down into five year periods and if the world is subdivided into regions as follows:

| Region | Percent Growth | | |
|---|---|---|---|
| | 1950-55 | 1960-65 | 1975-80 |
| World | 1.77 | 1.99 | 1.81 |
| More developed countries | 1.28 | 1.19 | 0.67 |
| Less developed countries | 2.00 | 2.35 | 2.21 |
| Africa | 2.16 | 2.49 | 2.91 |
| Latin America | 2.72 | 2.77 | 2.66 |
| Asia | 1.88 | 2.06 | 1.37 |

The world population in 1980 was approximately 4,500,000,000 people. Using a simple discrete exponential growth model we can make a reasonable prediction about the growth of this population over the next 10 years. We will use as our yearly growth rate the average yearly percent growth rate during the period 1975-1980, 1.81% per year. We will assume that this growth rate will remain valid during each of the next ten years. Under this assumption, the population of the world on Jan. 1 of each year is simply (1 + growth rate) times the population during the previous year.

| Year | Population on Jan. 1 | Increase during year (.0181 x population) |
|---|---|---|
| 1980 | 4,500,000,000 | 81,450,000 |
| 1981 | 4,581,450,000 | 82,924,000 |
| 1982 | 4,664,374,000 | 84,425,000 |
| 1983 | 4,748,799,000 | 85,953,000 |
| 1984 | 4,834,752,000 | 87,509,000 |
| 1985 | 4,922,314,000 | 89,094,000 |
| 1986 | 5,011,408,000 | 90,706,000 |
| 1987 | 5,102,114,000 | 92,348,000 |
| 1988 | 5,194,462,000 | 94,020,000 |
| 1989 | 5,288,426,000 | 95,722,000 |

**A Warning on the Use of Average Growth Rates**

When demographers present an average yearly growth rate, $R_{av}$, they generally obtain it by averaging over a five-year growth period. This is done as follows:

$$R_{av} = \frac{\text{Population now - Population 5 years ago}}{\text{Population 5 years ago x 5 years}}$$

Thus, a measured long term behavior (5 years) is used to calculate a short term behavior. We did exactly this above, when we used the 1.81% average yearly growth rate calculated over a 5 year period, to obtain projected yearly growths. It might seem that there is nothing wrong with using an average rate as an exact rate, but, in fact, it can lead to a slight discrepancy in calculations.

If a population, P, grows for five years at an *exact* yearly growth rate of 1.81%, the population grows to:

$$(1+0.0181)^5 P = 1.093\ P$$

Using this result, we can compute the average yearly growth rate during that period:

$$(1.093\ P - P)/(5\ P) = 0.093/5 = 0.0186 = 1.86\%$$

Thus, a constant yearly growth rate of 1.81%, applied for 5 successive years, does not produce an amount of growth corresponding to an average yearly growth rate of 1.81% during that period. The average growth rate is greater than the yearly rate because of the compounding effect of exponential growth. Conversely, when 5-year increases are used to calculate annual rates (as they are!), do not expect those annual rates to recreate the same 5 years of growth that you started with. Using averages can be a tricky business.

**PROBLEM 7**: The populations in the table above were calculated using a discrete model of exponential growth with a growth rate of .0181 per year. Calculate an analogous set of figures using a continuous exponential growth model and the same yearly growth rate.

**PROBLEM 8:** The 1986 population of the African nation of Rwanda is 6.5 million people. Its annual percentage growth rate is 3.8%. The area of Rwanda is 10,000 sq. mi. What is the population density of Rwanda (people per sq. mi.)? What is the density per sq. ft.? At the present growth rate and using a continuous model, in how many years would Rwanda have 1 person per sq. ft.?

**PROBLEM 9:** Using the growth rates given below and a continuous model of exponential growth, calculate the doubling times for the indicated regions and fill in the table below.

| Region | Percent Growth (1975-80) | Doubling time |
|---|---|---|
| World | 1.81 | |
| More developed countries | 0.67 | |
| Less developed countries | 2.21 | |
| Africa | 2.91 | |
| Latin America | 2.66 | |
| Asia | 1.37 | |

Using historical data we can examine world population growth over a longer period of time. The graph in Fig. 4 illustrates world population between 1650 and 2000. The last fifty-year period is based on an estimate using our growth model.

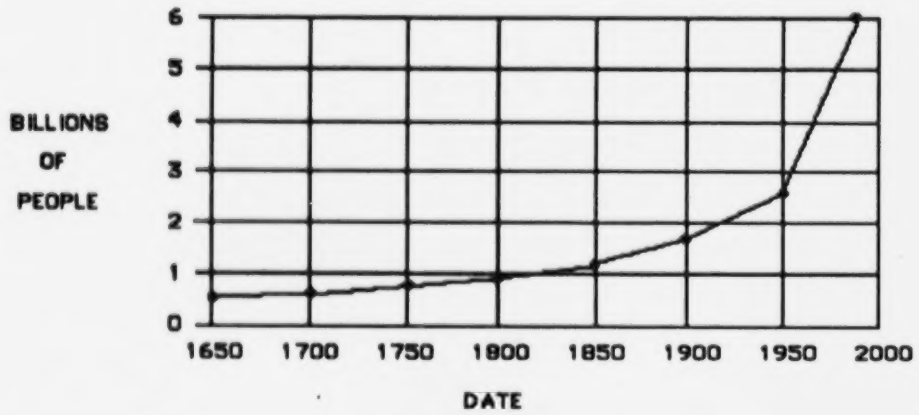



Fig. 4

If we trust our model enough to project exponential growth many decades into the future, we can arrive at the world population in 2800 AD, when the world will be nearly three deep with people! This catastrophe is depicted graphically in Fig. 5.

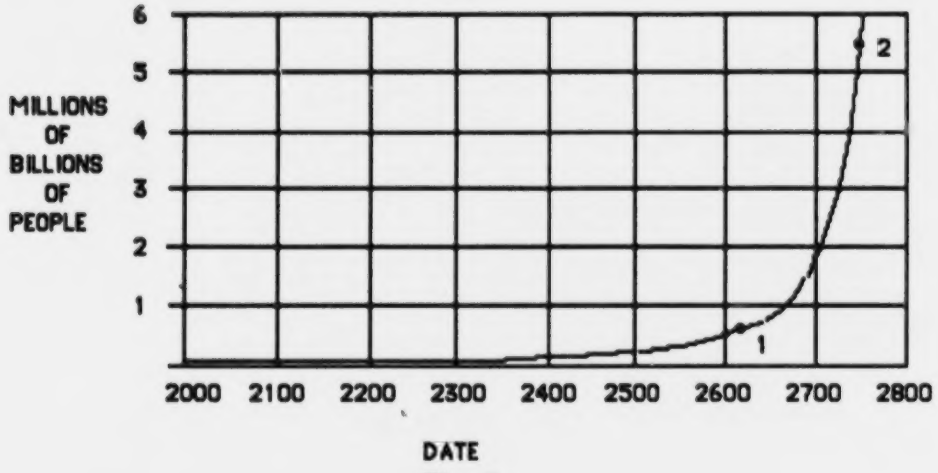



Fig. 5

**PROBLEM 10:** The diameter of the earth is 7920 miles. How many people would there be on the earth if one person stood in every square foot of total surface area (water included)?

In the graph of Fig. 5 the points labelled 1 and 2 represent one person for every square yard and one person for every square foot. Of course, there is probably good reason to suppose that the exponential growth model will cease being accurate well before these population densities are reached. Nevertheless, it is a sobering experience to examine the predictions of this model and see the potential for catastrophe that exists if our present growth rates go unchecked.

The present rate of increase of populations is obtained by collecting census data of population sizes over a period of years. Other kinds of data are also interesting and important and give insight into the reasons for exponential growth. Demographers are particularly interested in what they call the *replacement ratio* of a population. This number tells how many girl children a single girl child today will be replaced by in the next generation. The *fertility rate,* refers to the number of children of both sexes produced by a girl and is approximately twice the replacement ratio. In the United States this replacement ratio is 0.9, which is insufficient to maintain the U. S. population in the long run. Nevertheless, the U. S. population is still increasing even though it is not replacing itself (not a contradiction!) because today a disproportionate fraction of the U. S. population happens to be of childbearing age. Thus, even though that portion of the population is not growing, the mortality rate of the older portion of the population does not balance the birth rate of the younger portion, so there is a net population growth. This phenomenon is called *population momentum* and it demonstrates how a mere examination of average growth rates can conceal trends in the underlying population that

will have a considerable affect on the validity of the model. Ultimately, the distribution of age groups will smooth out and then the 0.9 replacement ratio, if it continues, will show up as a 10 percent decrease in population per generation. Many of the underdeveloped countries have a replacement ratio of 2 or more, which means that their populations double every generation. A generation is a length of time equal to the average age of childbearing in a country (approximately 25 years). A country with a replacement ratio of 2 will double its population every 25 years, which implies a growth rate of:

$R = 70/25 = 2.8$ percent

A country that doubles its population in 25 years will multiply it by a factor of 16 in a century!

**PROBLEM 11:** Show that a 2.8 percent growth rate in a continuous model leads to a multiplicative factor of just over 16 in a century.

For an extremely lucid and detailed introduction to the complexities of demography, see "Environmental Science," by Turk, Turk and Wittes (see bibliography), Chapter 4. Very complete and up-to-date references on world and individual country populations and growth rates can be obtained from the Population Reference Bureau Inc., 777 14th St. N.W., Suite 800, Washington D. C. 20005.

## SOLID WASTE DISPOSAL

Garbage, or as it is more euphemistically called, solid waste, is being produced at an alarming rate in the United States. In 1920 the average American created 2.75 pounds of solid waste per day. In 1970 the amount was 5 pounds per day and in 1980 approximately 8 pounds per day. The cost of collecting our residential, institutional and commercial waste ($3.5 billion in 1969) is more than the annual budget of the United Nations and many individual countries.

The problem of disposing of these wastes has reached immense proportions. Approximately 77 percent of them are placed in open dumps, 10 percent is incinerated and 13 percent is put into sanitary landfills. Unfortunately, in many areas the dumps are simply being filled up and the landfills (which are covered by earth to speed up bio-degradation) can't handle the high rates of dumping. Where shall the excess waste be placed? Regions fortunate enough to have unused open areas do not want to contribute them for use as dumps and landfills. Regions already having operating dumps and landfills, seeing them fill up and become health hazards, are restricting their use and, in many cases, closing them entirely. If you read a local paper you have already read about "illegal dumping operations" and fights between

townships over the rights to use dumping facilities.

**PROBLEM 12**: Investigate the solid waste disposal problem in your area. How much waste is being produced, what kind is it and what can you say about its rate of production? What kinds of local facilities are there for disposing of waste? Can you find out how fast these facilities are being used up? How many new facilities would be needed to properly handle the waste that is being produced?

**PROBLEM 13**: Non-returnable glass bottles and metal cans contribute heavily to the waste disposal problem. According to "The Complete Ecology Fact Book," the numbers (in millions of units) of non-returnable glass beverage containers, beer cans and soft drink cans used between 1955 and 1976 are the following:

| YEAR | SOFT DRINK CANS | BEER CANS | SOFT DRINK BOTTLES |
|---|---|---|---|
| 1955 | | | 169 |
| 1956 | | | 171 |
| 1957 | | | 183 |
| 1958 | 409 | 8337 | 192 |
| 1959 | 548 | 9156 | 211 |
| 1960 | 812 | 8888 | 249 |
| 1961 | 1225 | 8761 | 402 |
| 1962 | 1651 | 9075 | 477 |
| 1963 | 2058 | 9324 | 560 |
| 1964 | 2795 | 10896 | 632 |
| 1965 | 3841 | 11382 | 1006 |
| 1966 | 5612 | 12947 | 1980 |
| 1970 | 9000 | 15100 | 6000 |
| 1973 | 12300 | 16930 | 9000 |
| 1976 | 17000 | 19000 | 13500 |

Make graphs of these quantities, plotting number of units on the y-axis and the year on the x-axis. Do you think the growth in the various kinds of containers is best described by a linear or an exponential model? Find a growth rate that best describes each process (use the continuous model of exponential growth if it is applicable). Can you explain the increasing numbers of containers in terms of our increasing population during the same period?

## DEPLETION OF RESOURCES

The resources of the planet earth can be classed as either renewable or non-renewable. In the former class are such things as trees, food and water power. The class of non-renewable resources can be further broken down into those which are recoverable (i.e. as aluminum can be recovered from aluminum cans, etc.) and those which are not recoverable, such as coal, oil and other minerals. Since the earth is of finite size, it is clear that these non-renewable and non-recoverable resources are here in a finite supply.

Careful observation over many years demonstrates that the demand for and consumption of many non-renewable resources grows at a rate proportional to the rate of the population that is using the resource. Indeed, in many instances demand grows faster than the population because the same people want even more of the resource. Therefore, because the population is growing exponentially, the consumption of a resource like oil is also growing exponentially and, at some time in the future, the finite supply of oil must be depleted.

Several factors make it difficult to pinpoint exactly when the world's supply of crude oil will run out. 1. We don't know exactly how much oil there really is. We know what the current reserves are and we have good estimates from geological data of amounts that are potentially there to be discovered, but there are no "hard" numbers. 2. The growth rate of the population is decreasing and this will have a long-term impact on the consumption of oil. 3. Economic factors and conservation measures are reducing the per capita consumption of oil, so the consumption rate is dropping even without a decreasing rate of population growth. 4. Science and technology can be counted on to provide us with innovations that will reduce our dependency on oil. Certainly, if solar power is more fully developed, if fusion power comes online or if the problems associated with nuclear fission can be ameliorated, our consumption of oil should decrease. Even with all these factors clouding the future validity of the exponential growth model, however, it is undeniable that it has held true for 100 years and it is not too unreasonable to assume it will remain a viable model for the next 20 to 50 years. Therefore, we will use the model now and draw some sobering conclusions from it; but we will keep in mind that we must be ready to abandon the model as soon as its usefulness is in question.

According to data compiled by Dr. M. King Hubbert, a noted geophysicist from the U. S. Geological Survey (this data appears in the Bartlett paper, see references), the history of world crude oil consumption (also production) between 1880 and 1970 can be described by an exponential growth model with a 7 percent growth rate and a consequent doubling time of

$T_2 = 70/(7 \text{ percent per year}) = 10 \text{ years}$

This is a much faster doubling time than that of the population as a whole during that period (which was on the order of 40 years), indicating growth of demand even within the same population. The 10-year doubling time means that *in the next 10 years we will consume as much oil as we consumed in the entire period between 1880 and today!* This rather shocking conclusion, in turn, means that there must be sufficient oil reserves to supply all this oil if our demands are to be met. Stated another way, if a new source of oil were discovered tomorrow that was equal in size to all the oil we have ever consumed, that new supply would only last an additional 10 years! Many people have suggested that an enormous amount of oil resides in shale that is located in many parts of the U. S. If an amount of oil equal to what we have already consumed could even be extracted from shale, the left-over shale, which expands to many times its volume when the oil is removed, would probably cover most of the United States.

The present oil situation shows a distinct drop in the use of oil over the last 10 years. This is not a natural result of depletion, however, but an artificial result of the pricing policies of the OPEC nations. Recently, however, the price has begun to drop markedly, heralding, perhaps, another increase in the growth rate of oil consumption. However complex the political and economic situation may be, however, the finiteness of the world's supply of oil is not subject to doubt.

It is instructive to examine some of the data on crude oil production and consumption in the U. S. and the entire world.

**UNITED STATES CRUDE OIL DATA**
(See Bartlett and Almanac)

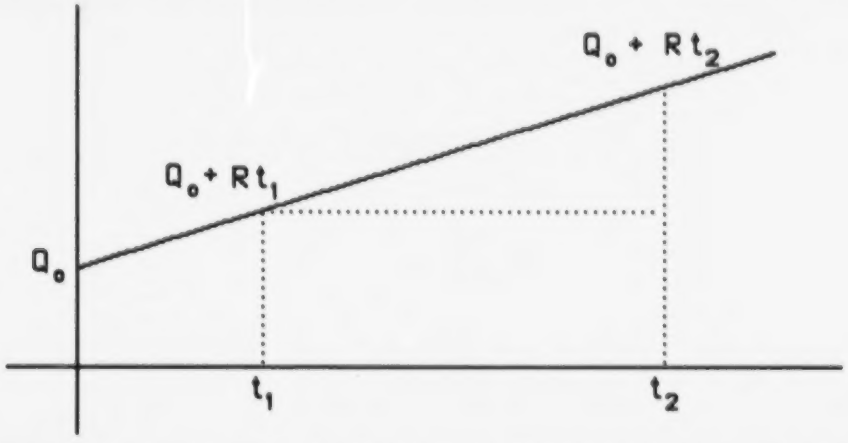
| | |
|---|---|
| Ultimate Total Production | 190,000,000,000 barrels |
| Produced To 1972 | 96,600,000,000<br>(50.8 percent of ultimate) |
| Annual Production Rates | |
| 1970 | 3,290,000,000 bbl/yr |
| 1979 | 3,121,000,000 bbl/yr |
| 1980 | 3,146,000,000 bbl/yr |
| 1981 | 3,125,000,000 bbl/yr |
| Annual Consumption | |
| 1979 | 6,757,000,000 barrels |
| 1980 | 6,242,000,000 barrels |
| 1981 | 5,840,000,000 barrels |

Note: the difference between production and consumption is made up by oil imports.

Up until 1970, the U. S. production of crude oil followed an exponential growth model with an annual growth rate of 8.27 percent. From 1979 on, the production rate levelled off to a nearly constant average rate of approximately 3,130,000,000 barrels per year. Thus, from 1979 to 1985 the assumption of exponential growth, or even linear growth for that matter, does *not* hold true!

## THE CUMULATIVE EFFECTS OF EXPONENTIAL GROWTH

When the quantity that is growing is the *production* or *consumption* of a resource, it is often necessary to know the total amount that has been produced or consumed during a certain period. This requires that the amounts given by the growth curve be summed over the period of interest. Obtaining such a sum is equivalent to computing the area under the production (or consumption) growth curve between $t_1$ and $t_2$, the two times that define the interval. If the curve is linear (linear growth), the area is quite easy to compute, because it is the sum of a triangular and rectangular area .




The total amount produced (or consumed ) during a period of linear growth of production (or consumption) is:

$$Q_0(t_2 - t_1) + 1/2\ R\ (t_2^{\ 2} - t_1^{\ 2})$$

Where $Q_0$ is the production or consumption at $t = 0$ and R is the growth rate of that production or consumption.

If the growth model is continuous exponential growth, the total amount of the quantity produced between $t_1$ and $t_2$ is also the area under the exponential curve, but it is not easily calculated from geometric considerations. It must be obtained by integration. We will simply write down the results of that integral:

$(Q_0/R)[e^{Rt_2} - e^{Rt_1}]$.

In the above expression $Q_0$ is the initial production or consumption at t=0 and R is the growth rate of that production or consumption.

**PROBLEM 14:** On the assumption (not presently correct) that U. S. oil production continued a 8.27 percent continuous exponential growth from 1970 onward, how long would it take to deplete its remaining reserves?

**PROBLEM 15:** On the assumption that the U. S. produced crude oil at the *constant* rate of 3,130,000,000 barrels/year from 1970 on, how long would it take to deplete its remaining reserves?

**PROBLEM 16:** On the assumption that the U. S. continued to produce oil from 1970 on according to the exponential model with the same annual growth rate associated with present world population growth , how long would it take to deplete present resources?

**PROBLEM 17:** Assume the Malthusian doctrine, that U. S. consumption of oil grows exponentially according to the present world population growth rate, but that through effective oil exploration methods we can add to our ultimate total production capacity at the linear rate of 3,000,000,000 barrels per year (the amount we actually presently produce per year). Under this assumption, when would we run out of oil?

WORLD CRUDE OIL DATA

| | |
|---|---|
| Ultimate Total Production | 1,952,000,000,000 barrels |
| Produced To 1972 | 261,000,000,000 barrels |
| Annual Production Rate in 1970 | 16,700,000,000 bbl/yr. |

The world production of crude oil has also been shown to follow an exponential growth model between 1880 and 1972 with a percentage growth rate of approximately 7 percent per year.

**PROBLEM 18 :** Assuming the exponential growth rate of world production holds true from 1972 on, approximately how long would its supply of oil last?

**PROBLEM 19:** Many of the arguments over the time it would take to deplete the entire world's oil resources result from the differing estimates of

exactly how much oil is still undiscovered. Bartlett provides an amazingly simple way of producing an absolute upper limit on the time remaining. He assumes the *entire earth* is a sphere of oil! The total volume of the earth would be equivalent to $6.81 \times 10^{21}$ barrels of oil—that is 6,810,000,000,000,000,000,000 barrels of oil. Taking this figure as the ultimate total production of world oil and using the 7 percent exponential growth rate from 1972 on, estimate how much time remains until we run out of oil. If anyone tells you we can find more oil after that, don't you believe it!

**Some Recent Comments on the Rate of Oil Depletion**

It seems appropriate to end this section and begin the following one with reference to a rather recent editorial from the journal *Science*, dealing with the future of U. S. oil production. (*Science*, vol. 226, pg. 426, 1984).

## LIMITS TO EXPONENTIAL GROWTH

In 1972, 174 years after Malthus published his essay, a group of eminent scientists calling themselves the "Club of Rome" published the results of a study in which they used sophisticated computer techniques to model the "predicament of mankind" resulting from the growth (or depletion) of such critical quantities as population, industrial production, fertilizer, energy, food, land, mineral reserves and pollution. They treated the problem as one that could be successfully modelled by taking into account all of the complex interactions between the various growth and depletion processes. In a lengthy report called "Limits To Growth," they conclude that "the basic behavior mode of the world system is exponential growth of population and capital, followed by collapse." Not surprisingly, in light of the reception that Malthus received, the entire report of the Club of Rome was subject to intense criticism. While there was some degree of criticism of the simplicity of their mathematical model, a great amount of criticism was aimed at their lack of faith in the ingenuity of man to avoid or overcome the very growth tendencies that they assumed would remain valid.

It is not proper to go into all their mathematical and sociological assumptions here, but they did derive some basic mathematical results that are perfectly valid and important for us to know and understand. In all our discussion so far, we have assumed that the exponential growth would continue unchecked. That is why it is called an exponential growth model. There are no other assumptions in it. In the more complex model developed by the Club of Rome, they include the supposition that as a resource is depleted, the amount still remaining in the ground requires extraction techniques that become increasingly expensive. We see this already in the case of oil. First we tap shallow underground beds. Then we dig deeper wells. Then we recover small amounts remaining in tapped wells by injecting high pressure steam into the wells. Finally, we are forced to extract oil from shale.

When the effect of cost on the extraction of a dwindling resource is in-

cluded in the model of the growth of the demand for that resource, the new model is no longer pure exponential growth, but a kind of inhibited exponential growth. The new growth curve is drawn rather roughly in Fig. 6 where we show not only the growth curve for the demand of the resource, but also the growth curve for the amount of resource remaining and the growth curve for the cost of extracting the resource.

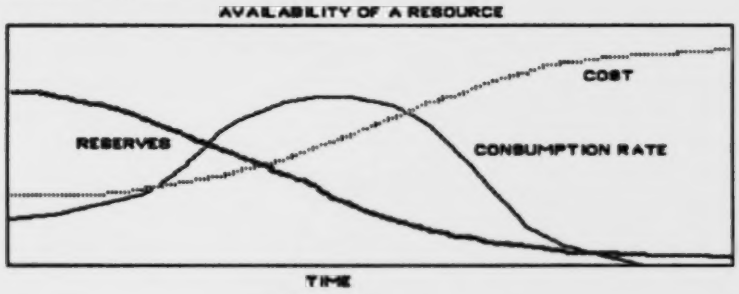



Fig. 6

It is easy to explain the shape of the demand growth curve in qualitative terms. The first part of the curve is exponential. In this region the demand grows with the number of people in the population who want the resource. If it is a resource that everyone ultimately wants, the demand will grow with the growth of the entire population or even faster than the population as individuals want more and more of it. The finiteness of the resource has not yet been felt. In the middle region, increasing cost of production has led to conservation measures and the substitution of alternative resources so the demand for the resources levels off. In addition, we begin to see saturation of the market. By and large, the number of people using the resource begins to level off. In the last part of the curve, we begin depleting the resource to the point where technology is inadequate to extract the remaining reserves. Even when we do extract them, the last dregs are usually lower quality and require higher refining costs, which in turn raises the overall price of the commodity.

M. King Hubbert was one of the first to note that exponential growth of the demand for a finite resource soon saturates and then diminishes to zero. He applied his analysis to fossil fuel consumption and drew a curve very similar to that produced by the Club of Rome. Hubbert's curve has come to be known as "Hubbert's Pimple," in recognition of its shape and its small size in the overall scheme of things.

The sketch of Hubbert's Pimple in Fig. 7 below is redrawn from Bartlett. It depicts the rather dismaying fact that all of the earth's supply of fossil fuels, created and deposited over a period of hundreds of thousands of years, will be totally depleted by us within a time span of a few hundred years. The entire idea that fossil fuels can be used to supply power may be relegated to a historical footnote.

Nearly all growth processes that are modelled by exponential growth are in

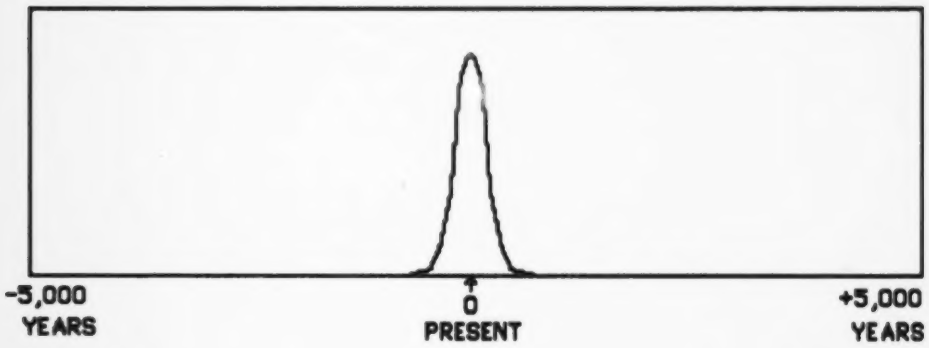



Fig. 7

an initial stage of their growth. Very few quantities can grow exponentially for long times because they grow beyond the abilities of their environments to sustain them. In the next section, when we discuss the growth of bacteria, we will mention briefly how their environment can inhibit their uncontrolled growth.

The next step in describing and understanding growth is to develop models that are accurate after exponential growth has stopped. This is a difficult task because the way in which growth deviates from exponential is very dependent on the specific quantity that is growing. There does not appear to be as universal a model for these later stages of growth as exponential growth is for the initial stages.

Sometimes what appears to be a deviation from exponential growth is simply a change in the growth rate while retaining the exponential character. When this occurs, growth is described by a succession of different exponential periods. A recent article in the journal *Science*, ("What Has Happened to Productivity Growth?", Martin Neil Baily, *Science*, Vol. 234, pp. 443, 24 Oct. 1986) discusses the important U. S. economic problem of the declining productivity of our industries and tries to find reasons for it. The author notes that productivity grew at an annual rate of 1.75 percent per year from 1953-1968, but fell to 0.32 percent per year from 1973 to 1979. Sudden and unexpected drops like this have a serious effect on economic planning. The author points out that our budget deficit would not exist today if the earlier growth rate had been sustained. Many businesses no doubt made long range plans based on the 1.75 percent growth rate, only to see those plans suffer in light of the 0.32 percent growth rate.

The growth in the use of certain technologies also shows changes in growth rate while maintaining overall exponential growth. An excellent example is the use of chemotherapy in cancer treatment. Chemotherapy was originally heralded as being effective in treating all forms of cancer. Its initial use, therefore, was among the entire group of cancer patients and the growth of this medical technology was exponential with a certain growth rate. As doctors gained more experience with its use, however, they realized that the

maximum effectiveness of chemotherapy was in dealing with certain types of cancers, thus the rate of use changed as the target group changed. Other growth processes exhibit changes that are not adequately described by simply changing the rate in an exponential model. These processes are beyond the scope of this module.

## OTHER QUANTITIES DESCRIBED BY AN EXPONENTIAL GROWTH MODEL

Populations and demand for resources seem to be the most important quantities associated with the exponential growth model. Populations grow exponentially primarily for complex biological and sociological reasons and resource demand and production is basically in response to population growth. Both of these growth processes have time as the independent variable.

The fable of the chessboard also illustrated exponential growth, but the underlying variable was not time, it was "square number", essentially a spatial variable. Another process described very accurately by an exponential growth model is of great importance to nuclear physicists; the decay of a quantity of radioactive material. A piece of radioactive material consists of a very large number of atoms, the number being symbolized by the letter N. The nuclei of these atoms are unstable, which means that every nucleus has a small probability of "decaying" in a certain unit of time; this decay probability is constant, which means a nucleus has the same chance of decaying regardless of how old it already is. The decay probability is symbolized by the Greek letter $\lambda$. When a nucleus decays, some particle in the nucleus, a proton or neutron, changes into another particle. As a result of this change, the original atom becomes a different atom. Since each undecayed atom has the same decay probability and since each atom decays independently of all other atoms (we assume that the decay of a particular atom in no way affects any of the other atoms), the total number of atoms that decay per unit time is simply $\lambda N$. Stated succinctly, total number decaying per unit time= probability of a single atom decaying per unit time, multiplied by the total number of atoms. This, however, defines exponential growth or, in this case, the exponential depletion of the original supply of atoms. Because the number of atoms is decreasing, the "growth" rate is negative and the expression for the number of undecayed atoms , N (t), at any time is:

$$N(t) = N_0 e^{-\lambda t}$$

where $N_0$ is the number at some initial time.

Just as we use the descriptive term "doubling time" in a growth process, we use the term *half life* to define the amount of time necessary to reduce the amount of radioactive material by 0.5. The half life, symbolized by the Greek symbol $\tau_{1/2}$, is defined by the same equation as doubling time:

$\tau_{1/2} = 0.7 / \lambda$

The quantity $\lambda N$, which is the number of atoms that decay in the unit time is called the *total activity* of the material at that time. Total activity is usually measured with a Geiger counter, an instrument that is sensitive to certain kinds of particles that the nucleus emits whenever the decay occurs. Because radioactivity is so important in many areas, it is assigned a special unit for its measurement. This unit is the *Curie*, abbreviated Ci. One Curie is defined as the quantity of radioactive material in which the total activity is $3.7 \times 10^{10}$ decays per second.

As the original collection of atoms disappears by decay, a new collection of atoms is growing: the atoms having the new nucleus resulting from the decay process. These are called the *daughter atoms*, the original atoms being the *parent atoms*. Since $\lambda N$ is the number of parent atoms decaying per unit time, it is also the number of daughter atoms created per unit time. Often a daughter atom also decays due to some decay process in its nucleus. This will usually give rise to a different decay probability, $\lambda'$, for the daughter. Thus the net increase in daughter atoms per unit time is the amount produced by the parent, minus the amount lost by decay of the daughter. This is a more complex process then simple exponential decay. The entire parent-daughter process is called a *radioactive series*.

**PROBLEM 20:** Compute the number of atoms in one millicurie (0.001 ci) of radioactive sodium ($Na^{24}$) whose halflife is 14.8 hours, radioactive phosphorus ($P^{32}$), halflife of 14.5 days and radium ($Ra^{226}$) with a halflife of 1,620 years.

**PROBLEM 21**: A sample of radioactive copper ($Cu^{64}$), halflife of 12.8 hours, is found to be undergoing $7.14 \times 10^4$ decays per second. How many curies are present? The copper is considered safe to handle in quantities of less than a microcurie (0.000001 ci). How long would you have to wait for the sample to become safe to handle?

**PROBLEM 22**: Radioactive iodine ($I^{130}$) decays with a halflife of 12.6 hours. Each decaying atom emits an electron with an energy of 0.29 MeV (million electron volts). An electron volt is an amount of energy equal to $1.6 \times 10^{-19}$ Joules ). In treating thyroid problems, it is not unusual to place 1 millicurie of $I^{130}$ in the thyroid for 24 hours. How much energy, in electron volts, is released into the thyroid during this period?

## BACTERIAL AND CELL GROWTH

The initial growth phase of a colony of bacteria (or a colony of cells in a biological organism) supplied with an appropriate environment can be described very well by exponential growth. Each bacterium divides to form two bacteria after the passage of a certain amount of time, g, called the *genera-*

*tion time*. The generation time is different for different bacteria and also depends on the environmental conditions under which the bacteria are being grown.

After n generation times have elapsed, which is ng seconds, an original single bacterium will have doubled n times and formed a colony of $2^n$ bacteria. We see, therefore, that this growth can be described very well by a discrete exponential growth model with the doubling time equal to the generation time.

In actuality, it is unrealistic to think of a bacteria colony as doubling precisely and synchronously as a unit every g seconds. If, indeed, you could start a colony from a single bacterium and keep each succeeding generation under the same precise environmental conditions, you might achieve this precise synchronized behavior. A biologist, however, usually starts out with a colony that has come from many different initial bacteria, so the members of the colony are splitting at random times. Under these less precise conditions, a continuous exponential growth model is more appropriate.

To use the continuous model, we need to express the bacterial growth in terms of a rate rather than a generation time. The generation time is a real biological property of a single bacterium. It is the length of time that it lives before dividing. A growth rate, on the other hand, is a property of a large group of bacteria. No single bacterium divides at a rate; it either divides or it doesn't. We must, therefore, go from a quantity that describes how an individual behaves to a quantity that describes how a group behaves. There is actually more than one way to do this and the choice is not merely for mathematical convenience, but depends on the way in which the colony is being treated. One approach is to assume that a single bacterium could live an arbitrary length of time before splitting, but that a large group of bacteria, on the average, will be found to split after a time g. This approach is completely analogous to that taken in describing radioactive decay. It can be shown (although we will not do so in this module) that under these conditions the rate of splitting, R, is simply:

$$R = 1/g$$

With this result, the number of bacteria, B, at any time t can be given in terms of $B_0$, the number at any initial time, as:

$$B = B_0 e^{t/g}$$

Note that this model leads to a doubling time:

$$T_2 = 0.7/R = 0.7/(1/g) = 0.7g,$$

which is less than the doubling time in the discrete model.

Bacteria are very sensitive to their environment. As the colony grows exponentially, eventually the environment is unable to supply it sufficient food or dispose of the waste products produced by the colony. When this point is reached, the exponential growth is inhibited and the size of the colony is stabilized. The point is called the *carrying capacity* of the environment. The resulting levelling off of the growth curve leads to a characteristic shape that is called *sigmoid*, meaning "shaped like an S." A sigmoid growth curve is shown in Fig. 8. It should be noted that Hubbert's Pimple is not a sigmoid curve because the earth's carrying capacity for petroleum use has been exceeded. The use of oil has not levelled off to take into account our finite oil supply; rather, we keep using it at a rate that leads to depletion rather than stability.

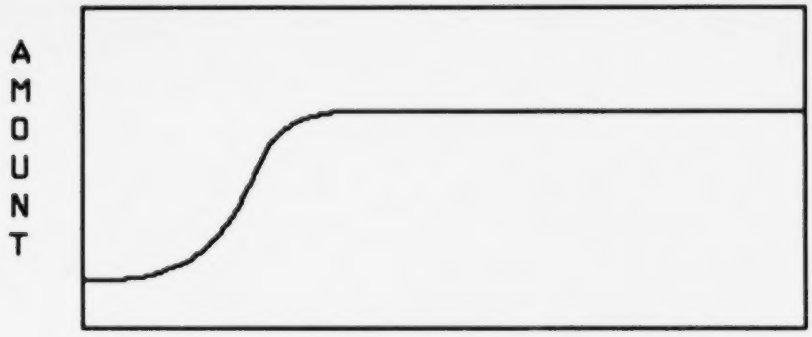



Fig. 8—Sigmoid Growth Curve

**PROBLEM 23:** We took a particular approach in passing from the discrete model to the continuous model which led to a shorter doubling time in the continuous model. It is also possible to simply pass to a continuous model that has the same doubling time as the discrete model. What would the growth rate have to be in the continuous model so that its doubling time is g, the same as in the discrete model?

**PROBLEM 24:** The bacterium E. Coli has a generation time of 20 minutes. Assuming a single E. Coli occupies an area 10 microns on a side ( 1 micron is 1 millionth of a meter ), how long would it take to form an E. Coli colony that would cover a circular dish 10 centimeters in diameter? Assume continuous growth with a doubling time of 0.7g and also assume that the bacteria spread out uniformly over the dish.

## EPIDEMICS

Epidemiology, the study of the spread of disease, is one of the most important branches of public health. The funding of medical research, the develop-

ment of immunization programs and the allocation of such hospital resources as beds, kidney dialysis machines and CAT scanners, is largely decided on the basis of the predictions made by epidemiological models.

While these models vary considerably in both mathematical and biological sophistication, one of the simplest is an exponential growth model. In this model we assume that the disease (presumed infectious) is contracted whenever an infected person comes in contact with a susceptible person. If every infected person comes in contact with, on the average, N susceptibles per unit time and if there are I(t) infected persons at time t, then the increase in the number of infected people is simply NI(t). Thus, the increase in infected people is proportional to the number that already exist; precisely the condition for exponential growth.

The formula for the number of infected people at time t is given by:

$$I(t) = I_0 e^{Nt}$$

where $I_0$ is the initial number of infected people (note, we are using the continuous model of exponential growth). This result predicts a doubling time for the number of infected people of:

$$T_2 = 0.7/N$$

Because we use the continuous model, we assume that infections are occurring on the finest possible time scale. It should also be noted that the model predicts non-integer values of the number of infected people, an insignificant shortcoming. The following, more significant, assumptions are also being made:

1. All infected people remain infected; they neither recover, die nor are quarantined.
2. The supply of susceptible people is unlimited.
3. Infections occur immediately on contact.
4. The infected people are dispersed throughout the population so that the constant rate of contacts, N, makes sense.

In more realistic models the above assumptions are modified, leading to non-exponential models.

**PROBLEM 25:** Compare the validity of the above assumptions for a disease like AIDS vs. a disease like influenza.

**PROBLEM 26:** The doubling time of AIDS between 1982 and 1986 has been approximately one year. Assume there were 1000 known cases in 1982, how long would it take for the entire world's population to become infected assuming the model remains valid. Do you think the model would break down before this happens? Why?

## THE NUCLEAR CHAIN REACTION

The nuclear chain reaction which is the heart of every nuclear fission reactor is another phenomenon described very accurately by a continuous exponential growth model.

The heat produced in the core of a nuclear reactor is the result of a basic nuclear reaction in which the nucleus of an atom of $^{235}U$ (Uranium, having an atomic mass of 235) absorbs an incident neutron and "fissions", splits into a variety of two smaller nuclei, emits two neutrons and a great deal of energy (as heat). Typically, a fission reaction is written:

$$U^{235} + n = Sr^{97} + Xe^{137} + n + n + \text{heat}$$

indicating that, in this case, the two smaller nuclei are Strontium and Xenon.

The two neutrons produced in this reaction can, in principle, initiate two further fissions and, thereby, produce four new neutrons. This would happen if each neutron struck another $U^{235}$ nucleus under the proper circumstances. Reactor engineers calculate a quantity, k, called the "neutron multiplication factor", which is the ratio of new neutrons to old. If the two neutrons did produce four, the multiplication factor would be two.

In order for the neutrons to initiate additional fissions, however, they must be slowed down enormously, from an initial speed of approximately $10^7$ meters/second to about 2200 meters/second. The reason for the necessity of slowing the neutrons is that the $U^{235}$ nucleus will not absorb a fast neutron, it literally bounces off. The slowing down is accomplished by a "moderator", like water, which surrounds the fuel elements and through which the neutrons must pass. The neutrons collide with the water molecules and slow down very quickly, in about 30 millionths of a second ( 30 microseconds, 30 x $10^{-6}$ seconds). The number of neutrons in the core of the reactor therefore multiplies by a factor of k every 30 microseconds, or approximately 30,000 times each second. In one second, therefore, the initial number of neutrons is multiplied by:

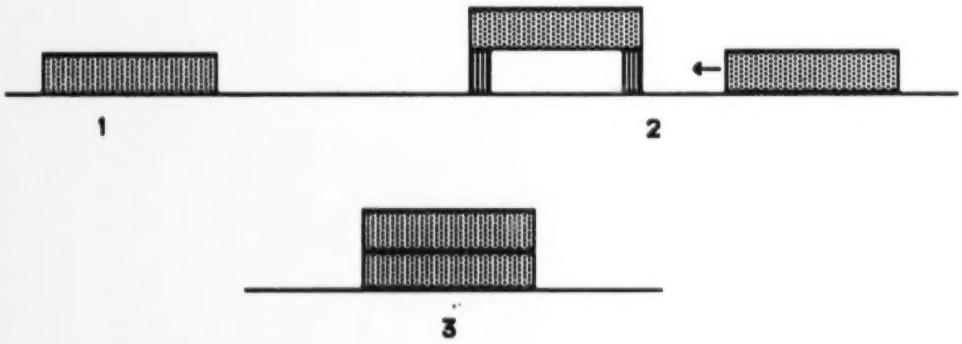
$$k^{30,000}.$$

Since each neutron in this enormous number has been created in a fission process, there is an enormous amount of heat being generated. If, indeed, k=2, the multiplication of neutrons and heat would probably lead to an uncontrollable reaction which would melt the reactor.

What saves all reactors from melting down is the fact that k does not equal 2, in fact it normally is only slightly greater than 1. This is because many neutrons never strike another nucleus, but either escape completely from the core of the reactor or are captured in special "absorber" rods, called control rods, that are inserted in the reactor between the fuel element assemblies. In an operating reactor the value of k is about 1.001, analogous to a

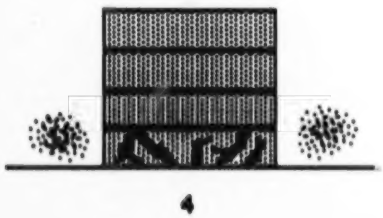
slow simmer. When k is less than 1, the reactor is called "subcritical" and will neither maintain a chain reaction nor generate power. When k exceeds 1, the reactor is "supercritical", an undesirable condition necessitating insertion of the control rods. A reasonable, self sustaining chain reaction involves a k of 1, called "criticality."

**PROBLEM 27:** Each fission produces 200 MeV ( million electron volts) of energy. 1 eV = $1.6 \times 10^{-19}$ J (Joules). How much energy is produced by $(1.001)^{90,000}$ fissions? If this much energy were produced each second, how many Watts of power would be generated? 1 Watt = 1 J/second




**BUILDING A TOWER**

Imagine you were going to build a tower by stacking cement blocks one on top of the other. Imagine also that your construction technique involved placing the first block on the ground, raising it up, placing the second block beneath the first, raising them both up and placing the third block beneath those two and so on.




As it turns out, however, cement blocks can only withstand a certain constant pressure ( pressure=weight they support/their surface area) before they crumble. If you simply keep piling blocks one on top of the other, the bottom block will eventually crumble and the tower will collapse, leading to a pile of rubble and many lawsuits.

One way to solve this problem is to keep increasing the area of each new block that is inserted under the present pile, so as to keep the pressure on it constant. Unfortunately, the weight of a block also depends on its area (if they all have the same thickness, T, and the density of concrete is d, then the weight of a block is ATd, where A is the area of the block). Thus, the problem of calculating what the area of each new bottom block should be is not simple.

Let us do this calculation in a crude way. Assume each block is capable of withstanding a pressure of P pounds per square foot. Let the first block have an area $A_1$ and a weight $W_1$, where $W_1 = TdA_1$. We will cast block 2 so that it has the same area as block 1 plus an additional area sufficient to withstand the weight of block 1. Thus:

$$A_2 = A_1 + (W_1/P) = A_1 + (TdA_1/P)$$

The weight of block 2 is, therefore:

$$W_2 = TdA_2$$

Block 3, therefore, must have an area equal to the basic area of block 1, plus a sufficient additional area to support the weights of blocks 1 and 2 combined:

$$A_3 = A_1 + (W_1 + W_2)/P$$

$$A_3 = A_2 + dTA_2/P$$

Thus we see that the area of each new block equals the area of the previous block plus a fraction, dT/P, of that area. In short, the block area obeys a model of exponential growth. If the area grows exponentially and the block is square, then the width of the block also grows exponentially. A side view of the tower looks like the following.

5

Of course the design of real towers must take many more things into consideration. The shape of the Eiffel tower, for example, looks like an exponential curve, but actually it is a parabola. This shape, moreover, is not a result of the weight it must support, but it is necessary for the tower to withstand

the force of wind against it. By and large we don't build towers of piled cement blocks these days, but the example is just meant to show the strange ways in which exponential growth model can appear in the world around us.

**PROBLEM 28:** You can really mystify your friends with this "trick." Ask someone to pick any word in a dictionary and offer to guess it in 20 questions or less. Almost certainly nobody will believe you can do it. Proceed in the following way. Divide the dictionary in half at its center page and ask: "Is the word between A and (whatever word ends the central page)?" If the answer is "yes," divide the first half of the dictionary in half and proceed in the same way. If the answer is "no," divide the second half of the dictionary in half and proceed in the same way. Continue dividing and determining which half contains the word. Eventually you will get down to one page, then one side of a page, then one column on a page and, finally, the chosen word. Analyze this amazing feat in terms of exponential "growth." How big would a dictionary have to be for twenty questions to be insufficient? Would this ever happen?

**PROJECT:** In this project a laboratory group will add water to a bucket at a fast enough rate to make the amount of water increase exponentially. This should give you an intuitive feel for exponential growth.

The equipment for this experiment is: a large bucket, an eyedropper, a stopwatch and a selection of various size containers.

The task is to put water into the bucket drop by drop so as to double the amount of water in it every 10 seconds.

Proceed as follows. Start your stopwatch and put one drop into the empty bucket. 10 seconds later put enough water in to double what is already there, i.e. add one more drop. Then, 10 seconds later add 2 drops, making a total of 4 drops in the bucket. Continue this procedure until the bucket is full. Measure how much time the filling takes. Your group will have to use all its ingenuity to maintain the constant doubling time. When you think you can do it, call over your lab instructor and have him/her watch.

## REFERENCES

Bartlett, Albert A. "Forgotten Fundamentals of the Energy Crisis," *American Journal of Physics* 46 ( Sepember 1978): 879.

This paper is a lucid exposition of the energy crisis, debunking many myths through simple use of mathematics and the exponential growth model. It also contains a listing of many useful references.

*Outlook for Science and Technology; A Report of the National Research Council* W. H. Freeman & Co., San Francisco, 1982.

A valuable source of information on trends in world population. While it does not deal with mathematical models, it does present the demographic reasoning that aids in the understanding and construction of models.

Meadows, Donella H., and Dennis L. Meadows, Jorgen Randers, William W. Behrens III. *The Limits To Growth.* New York: New American Library, 1972.

A kind of classic in terms of the mathematical modelling of population and resource dynamics using computers. Although criticized by many for varied reasons, it should be read by anyone interested in exponential growth.

*Man and His Technology*. Engineering Concepts Curriculum Project, SUNY. New York: McGraw Hill Book Co., 1971.

An excellent and still timely treatment of the role of models in many areas of technology with a particularly well written section on populations and exponential growth. Contains all the mathematics you will ever need.

Turk, Turk and Wittes. *Ecology, Pollution, Environment*. Phila.: W. B. Saunders Co., 1972.

Contains an excellent chapter on population growth; also discusses radioactivity.

*The Hammond Almanac, 1983*. Maplewood, NJ: Hammond Almanac Inc., 1983.

The almanac is always an excellent reference for up-to-date information and data points for your model.

Frauenthal, J. C. *Mathematical Modeling in Epidemiology*. Berlin: Springer-Verlag, 1980.

An excellent introduction to mathematical models, but requires an under standing of differential equations. There is good discussion of the medical and biological premises on which many models are based, however, and these can be read by anyone.

Levine, Melvin M. *Fission Reactors.* Published by the American Association of Physics Teachers, 1983.

An excellent teaching module for nuclear fission reactors. Everything you'll ever need to know, presented lucidly and with a minimum of mathematics.

Nobile, Philip and John Deedy, Eds. *The Complete Ecology Fact Book*. New York: Doubleday, 1972.

A bit dated, but contains a wealth of ecological facts. Of particular interest are such tidbits as a graph showing the correlation between the exponential growth of population and the extinction of various animal spe cies (pg. 106).

Moran, Morgan and Wiersma. *Introduction to Environmental Science.* New York: W.H. Freeman and Co., 1986.

Contains a good chapter on the biological regulation of population growth.

**APPENDIX 1**

**The Complete IRA Program**

```
180   '   * * * * * * * * * * * * * * * * * * *
190   '   * * *  Declaration of Variables  * * *
200   '   * * * * * * * * * * * * * * * * * * *
210   '
220   '   CLIENT$:  name of user, supplied as input
230   '   CONTRIBUTION:  VALUE invested each year, supplied as input
240   '   AGE:  age of user when IRA is opened, supplied as input
260   '   RATE:  interest rate
270   '   YEAR:  Current Year
280   '   RETIREMENT.AGE:  age at which client retires.
290   '   INVESTMENT:  Total contributed to IRA
300   '   VALUE:  Value of IRA
310   '
320   '   * * * * * * * * * * * * * * * * * * * * * * * * *
330   '   * * *  INITIALIZATION OF VARIABLES  * * *
340   '   * * * * * * * * * * * * * * * * * * * * * * * * *
350   '
360   LET YEAR = 1985                  '   Modify to current year
370   LET RETIREMENT.AGE = 60  '   Modify if desired
380   ' here comes a LINE INPUT with a ? supplied by us
390   LINE INPUT "What is your name? "; CLIENT$
400   INPUT "How old are you"; AGE
410   PRINT "How much will you invest in your IRA each year";
415   INPUT CONTRIBUTION
420   INPUT  "What is the interest rate (as a percent)"; RATE
425   LET RATE = RATE/100              '   convert to decimal
430   LET VALUE = CONTRIBUTION
440   LET INVESTMENT = CONTRIBUTION
450   '
460   '   * * * * * * * * * * * * * * * * * *
470   '.  * * *  MAIN PROCESSING  * * *
480   '   * * * * * * * * * * * * * * * * * *
490   '
495   PRINT
500   PRINT " " ,  "IRA REPORT FOR ";CLIENT$
510   PRINT
520   PRINT "YEAR", "AGE", "INVESTED", "VALUE OF IRA"
530   PRINT
540   '
550   WHILE AGE < RETIREMENT.AGE
560      LET VALUE = VALUE * (1 +  RATE)
570      LET YEAR = YEAR + 1
580      LET AGE = AGE + 1
590      PRINT YEAR, AGE, INVESTMENT, VALUE
600      LET VALUE = VALUE + CONTRIBUTION
```

```
610       LET INVESTMENT = INVESTMENT + CONTRIBUTION
620    WEND
630    '
640    PRINT
650    PRINT "END OF IRA REPORT"
660    '
670    END
```

## PROBLEM SOLUTIONS

2. $[1 + (0.05/365)]^{365} = (1.000137)^{365} = 1.0513$

Therefore, $100.00 grows to $105.13

3. $3200.00

4. $e^{RT_3} = 3$

$RT_3 = \ln(3) = 1.098$
$T_3 = 1.098/R$, or:
$T_3 = T_2 ( \ln(3) / \ln(2) )$

5. Two centuries is 8 doubling times. If $P_0$ is the initial population and $F_0$ is the initial food supply, then the population after 8 doubling times is $2^8 P_0$ and the food supply is $P_0 + 8 P_0 = 9 P_0$. The ratio of population to food supply is $(256/9) P_0/F_0$.

7. For example, on Jan. 1, 1981, the population is:
$4{,}500{,}000{,}000\ e^{.0181} = 4{,}582{,}191{,}600.$

8. Population density= 650 people per sq. mi. = $2.33 \times 10^{-5}$ people per sq. ft.

To increase this to 1 person per sq. ft. requires multiplication by a factor of $42.9 \times 10^3$. Therefore:

$e^{.038t} = 42.9 \times 10^3$

$.038\ t = 10.66$

$t = 280.7$ years.

9.

| | |
|---|---|
| World doubling time | = 70/1.81 = 38.7 years |
| MDC doubling time | = 104.5 years |
| LDC doubling time | = 31.7 years |
| Africa doubling time | = 24 years |
| Latin America | = 26.3 years |
| Asia | = 51.1 years. |

10. Area of earth = $\pi d^2 = 1.97 \times 10^8$ sq. mi. = $5.48 \times 10^{15}$ sq. ft.

This would lead to a population of $5.48 \times 10^{15}$ people, which is 5.48 million billion people.

11. $e^{(.028)(100)} = 16.4$

14. Reserves remaining = Ultimate total production - Amount produced to 1972 = 93,400,000,000 barrels.

We want to find t, where:

$(Q_0/R)[e^{Rt} - 1] = 93,400,000,000.$
Since $Q_0$ is the 1970 production rate: 3,290,000,000 barrels per year and R=.0827, we find that t = 14.5 years.

15. t = 28.78 years.

16. t = 22.8 years.

17. This is very difficult to solve exactly. Use $6 \times 10^9$ barrels per year as our consumption rate in 1972 on and see how long it would take to consume our remaining reserves of $93.4 \times 10^9$ barrels at a 1.81% growth rate of that consumption. Then calculate how many additional barrels we would have added during that time at a rate of $3 \times 10^9$ barrels per year and calculate how long it would have taken to consume that. You can work back and forth to arrive at an approximate answer.

18. Done just like problem 16, but use world crude oil data on pg. 24.

19. t = 297.2 years.

20. $Na^{24}$: $\tau_{1/2} = .7/\lambda$. A half-life of 14.8 hours = $5.3 \times 10^4$ seconds.
Therefore $\lambda = 1.31 \times 10^{-5}$.
1 millicurie = $3.7 \times 10^7$ decays/s = $\lambda N$
$N = 2.82 \times 10^{12}$ atoms.
For $P^{32}$: $N = 68.7 \times 10^{12}$ atoms

For $Ra^{224}$: $N = 2.7 \times 10^{18}$ atoms.

21. There are 2 microcuries present. Waiting 12.8 hours will reduce this to a safe amount.

22. In 24 hours the 1 millicurie source has decayed to 1/2 millicuries. The number of atoms in 1 millicurie is:

$N = (3.7 \times 10^{10})/(1.54 \times 10^{-5}) = 2.4 \times 10^{15}$ atoms, in 1/2 millicurie it is half that amount.

Therefore, $1.2 \times 10^{15}$ atoms have decayed, emitting that many electrons and a total energy of: $0.348 \times 10^{15}$ MeV.

23. If $T_2 = g$ and $T_2 = .07/R$, then $R = 0.7/g$.

24. Area of one E. coli= 100 microns$^2$ = $10^{-10}$ m$^2$. The dish has an area of 0.0314 m$^2$. The number of E. coli that will cover the dish is $3.14 \times 10^8$. Therefore:

$10{,}000\ e^{t/g} = 3.14 \times 10^8$

t= 3.45 hours.

26. World population is approximately $6 \times 10^9$.
$6 \times 10^9 = 1000\ e^{Nt}$
$N = .7/T_2 = .7/1$ year. Therefore:
$6 \times 10^6 = e^t$
t = 15.6 years.

27. $(1.001)^{30{,}000} = 1.05 \times 10^{13}$
Energy = $(1.05 \times 10^{13})(200 \times 10^6 \times 1.6 \times 10^{-19}$ Joules)= 336 Joules
If this amount of energy is produced each second, it is 336 Watts.

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